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5478: *Proposed by D. M. Btinetu-Giurgiu, "Matei Basarab" National Collge, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzu, Romania*

Compute:

$$\int_0^{\pi/2} \cos^2 x \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx.$$

Solution 1 by Karl Havlak, Angelo State University, San Angelo, TX

Let $u = \cos x$. Then $du = -\sin x dx$ and $\sin(x) = \sqrt{1-u^2}$. We may rewrite the given integral as

$$\int_0^1 \left(u^2 \sin^2\left(\frac{\pi}{2}u\right) + \frac{u^3}{\sqrt{1-u^2}} \sin^2\left(\frac{\pi}{2}\sqrt{1-u^2}\right)\right) du.$$

Considering the second term in the integrand, we let $v = \sqrt{1-u^2}$ so that $dv = -\dfrac{u}{\sqrt{1-u^2}} du$ and $u^2 = 1 - v^2$. We may write the integral above as

$$\int_0^1 u^2 \sin^2\left(\frac{\pi}{2}u\right) du + \int_0^1 (1-v^2) \sin^2\left(\frac{\pi}{2}v\right) dv.$$

This reduces to $\displaystyle\int_0^1 \sin^2\left(\frac{\pi}{2}v\right) dv$, which can be easily shown to be $\frac{1}{2}$.

Solution 2 by Moti Levy, Rehovot, Israel)

$$\begin{aligned} I &:= \int_0^{\frac{\pi}{2}} (\cos^2 x) \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx \\ &= \int_0^{\frac{\pi}{2}} \cos^2 x \sin x \sin^2\left(\frac{\pi}{2}\cos x\right) dx + \int_0^{\frac{\pi}{2}} \cos^2 x \cos x \sin^2\left(\frac{\pi}{2}\sin x\right) dx \end{aligned}$$

Change of variables, $u = \cos x$ in the first integral and $v = \sin x$ in the second integral gives

$$\begin{aligned} I &= \int_0^1 u^2 \sin^2\left(\frac{\pi}{2}u\right) du + \int_0^1 (1-u^2) \sin^2\left(\frac{\pi}{2}u\right) du \\ &= \int_0^1 \sin^2\left(\frac{\pi}{2}u\right) du = \frac{1}{2}. \end{aligned}$$

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$I = \int_0^{\pi/2} \cos^2 x \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx$$

$$
\begin{aligned}
&= \int_0^{\pi/2} (1-\sin^2 x)\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \int_0^{\pi/2} \cos^3 x \sin^2\left(\frac{\pi}{2}\sin x\right) dx \\
&= I_1 - I_2 + I_3 \text{ where}
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \int_0^{\pi/2} \sin x \sin^2\left(\frac{\pi}{2}\cos x\right), \quad I_2 = \int_0^{\pi/2} \sin^3 x \sin^2\left(\frac{\pi}{2}\cos x\right) dx \text{ and} \\
I_3 &= \int_0^{\pi/2} \cos^3 x \sin^2\left(\frac{\pi}{2}\sin x\right) dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
I_1 &= \int_0^{\pi/2} \sin x \frac{1-\cos(\pi\cos x)}{2} dx = \int_0^{\pi/2} \frac{\sin x}{2} - \frac{\sin x(\cos\pi\cos x)}{2} dx \\
&= \left[-\frac{\cos x}{2} - \frac{\sin(\pi\cos x)}{2\pi}\right]_{x=0}^{x=\pi/2} \\
&= \frac{\cos(\pi/2)}{2} - \frac{\sin(\pi\cos(\pi/2))}{2\pi} - \left(-\frac{\cos 0}{2} - \frac{\sin(\pi\cos 0)}{2\pi}\right) \\
&= 0 - 0 + \frac{1}{2} + 0 = \frac{1}{2}.
\end{aligned}
$$

With the substitution $t = \frac{\pi}{2} - x$, one obtains that

$$
\begin{aligned}
I_2 &= \int_0^{\pi/2} \sin^3 x \sin^2\left(\frac{\pi}{2}\cos x\right) dx = \int_{\pi/2}^{0} \sin^3\left(\frac{\pi}{2} - t\right) \sin^2\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2} - t\right)\right)(-dt) \\
&= \int_0^{\pi/2} \cos^3 t \sin^2\left(\frac{\pi}{2}\sin t\right) dt = I_3.
\end{aligned}
$$

The value of the given integral is therefore $I = I_1 = \frac{1}{2}$.

Also solved by Yen Tung Chung, Taichung, Taiwan; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ravi Prakash, New Delhi, India; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Shivam Sharma, New Delhi, India; Albert Stadler, Herrliberg, Switzerland, and the proposers.